Pergamon

Int. J. Solids Structures Vol. 33, No. 19, pp. 2759–2774, 1996

0020-7683/96 $15.00 + .00

0020–7683(95)00181–6

CERTAIN TWO-PUNCH PROBLEMS FOR AN ELASTIC LAYER

Q. LAN,
Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, QC, Canada H3A 2K6

G. A. C. GRAHAM
Applied and Computational Mathematics Program, Simon Fraser University, Burnaby, BC, Canada V5A 1S6

and

A. P. S. SELVADURAI
Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, QC, Canada H3A 2K6

(*Received* 29 *October* 1994; *in revised form* 18 *July* 1995)

Abstract—This paper examines the normal and tangential indentation problems for two circular punches on the surface of an elastic layer. By using the method proposed for studying offset parallel penny-shaped cracks in an elastic solid, we show that these problems are governed by systems of Fredholm integral equations, which for some special cases, can be solved by iteration. For certain prescribed indentations, asymptotic solutions are presented to illustrate the manner in which the presence of the second punch and thickness of the layer influences the total forces induced on the punches. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Normal indentation of a half-space region by two circular punches has been considered by Collins (1963). In this study the problem is reduced to a system of Fredholm integral equations of the second kind, which are then solved iteratively for the case when the punches are far apart. By using Galin's expression for pressure under the punch caused by a concentrated normal load at another point of the half-space, Gladwell and Fabrikant (1982) derive simple approximate relationships among the forces, moments, and indentations for a system of circular punches on a half-space. These results are then extended by Fabrikant (1986) to include elliptic punches. A related problem, in which tangential displacements instead of normal displacements are prescribed in the contact regions, is first investigated by Fabrikant (1989). By using the mean value theorem, he relates the resulting tangential forces acting on each domain to the given displacements through a system of linear algebraic equations.

This paper deals with interaction of two circular indentors on an elastic layer. Both the normal indentation and a "relaxed" tangential indentation problems are examined. It is shown that these problems are governed by systems of Fredholm integral equations of the second kind, which can be solved approximately by iteration for the case where the radii of the contact areas are small when compared with the distance between them and thickness of the layer.

The basic methodology of the present analysis, which follows from the procedures given by Graham and Lan (1994) for studying offset cracks, is outlined in Section 2. Derivation of the governing integral equations for the normal indentation problem and complete details of their solutions are also presented in this section. The tangential indentation problem is solved in a very similar fashion with the aid of the method given by Westmann (1965) for solving simultaneous pairs of integral equations. For the sake of brevity, only some of the important formulae and results are presented in Section 3.

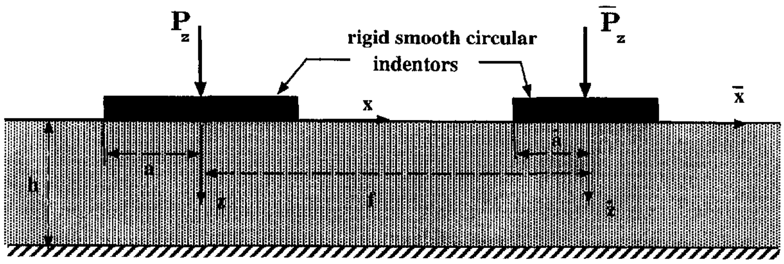

Fig. 1. Normal indentation of an elastic layer by two rigid circular smooth indentors.

2. NORMAL INDENTATION PROBLEM

The normal indentation problem can be described as follows: an elastic layer rests on a frictionless rigid foundation and the layer is indented by two lubricated circular rigid punches P and $\bar{P}$ of radii a and $\bar{a}$ respectively. We evaluate the forces P_z and $\bar{P}_z$ required to maintain a prescribed set of displacements of the punches.

Consider two similarly oriented local cylindrical co-ordinate systems (r, θ, z) and $(\bar{r}, \bar{\theta}, z)$ such that the surface contact areas occupy

$$P: r < a, \quad 0 \leqslant \theta \leqslant 2\pi, \quad z = 0; \quad \text{and} \quad \bar{P}: \bar{r} < \bar{a}, \quad 0 \leqslant \bar{\theta} \leqslant 2\pi, \quad z = 0, \tag{1}$$

respectively, and the elastic layer has a finite thickness h such that $0 \leqslant z \leqslant h$. The two coordinate systems are arranged in such a way that the origin $\bar{O}$ of the second coordinate system is a point $(f, 0, 0)$ in terms of the first set of coordinates, and O is a point $(f, \pi, 0)$ in terms of the second set. Here f is the distance between the centers of the two circular contact areas (Fig. 1). In this section we assume that displacement field of the elastic layer takes the following form*

$$u_r(r,\theta,z) = \sum_{n=0}^{\infty} u_r^n(r,z)\cos(n\theta); \quad u_\theta(r,\theta,z) = \sum_{n=1}^{\infty} u_\theta^n(r,z)\sin(n\theta);$$

$$u_z(r,\theta,z) = \sum_{n=0}^{\infty} u_z^n(r,z)\cos(n\theta), \tag{2}$$

where $u_r^n(r,z), u_\theta^n(r,z)$ and $u_z^n(r,z)$ are Fourier coefficients of the displacement vector. For such a displacement field, stresses can also be expanded as Fourier series. Let $\tau_{rz}^n(r,z), \tau_{\theta z}^n(r,z)$ and $\sigma_z^n(r,z)$ be the corresponding Fourier coefficients of the stresses of interest. In terms of these Fourier coefficients, boundary conditions for the normal indentation problem can be written as follows

$$\tau_{rz}^n(r,0) = \tau_{\theta z}^n(r,0) = 0, \quad \text{for } r \geqslant 0; \quad \sigma_z^n(r,0) = 0, \quad \text{for } r > a \quad \text{or} \quad \bar{r} > \bar{a};$$

$$u_z^n(r,0) = f_n(r) \quad \text{for } r \leqslant a; \quad \text{and} \quad u_z^n(\bar{r},0) = \bar{f}_n(\bar{r}) \quad \text{for} \quad \bar{r} \leqslant \bar{a} \tag{3}$$

on the indentation surface $z = 0$ and

$$u_z^n(r,h) = \tau_{rz}^n(r,h) = \tau_{\theta z}^n(r,h) = 0 \tag{4}$$

at the base of the layer $z = h$. The boundary conditions imply a frictionless bilateral contact with no separation at the interface $z = h$. Here $f_n(r)$ and $\bar{f}_n(\bar{r})$ are prescribed functions determined by the indentations and the profiles of the punches.

* If the dependence on θ (even or odd) of the loadings is changed, all the formulae remain valid except for some minor changes.

2.1 *Derivation of the integral equations*

In order to solve the two punch normal indentation problem, we first examine a more convenient layer problem. Consider the problem of an elastic layer ($0 \leqslant z \leqslant h$) which rests on a frictionless rigid foundation at a depth $z = h$ with prescribed normal stress applied at the upper surface $z = 0$. In terms of the cylindrical coordinates (r, θ, z), the boundary conditions of this layer problem can be written as

$$\sigma_z^n(r,0) = p_n(r), \quad \text{for } r \geqslant 0; \quad \tau_{rz}^n(r,0) = \tau_{\theta z}^n(r,0) = 0, \quad \text{for } r \geqslant 0 \tag{5}$$

on the upper surface $z = 0$, where $p_n(r)$ is the Fourier cosine coefficients of the given normal traction. At the base of the layer, boundary conditions (4) are imposed.

Solution for this layer problem can be obtained by using either the method proposed by Keer (1964) or by employing the general formulation given by Muki (1961) for the three-dimensional asymmetric problem. In deriving the integral equations for the two punch normal indentation problem, we only require the relationship between the normal displacement on the top surface and the given normal stress. In terms of the co-ordinates (r, θ, z), this relation takes the form

$$u_z^n(r,0) = -\frac{(1-\nu)}{\mu}\int_0^\infty A_n(s)[1-K_1(2sh)]J_n(rs)\,\mathrm{d}s \tag{6}$$

where ν, μ are Poisson's ratio and shear modulus of the elastic layer respectively, $J_n(rs)$ is the Bessel function of the first kind of order n, $A_n(s)$ is the nth order Hankel transform of the given normal stress Fourier coefficients $p_n(r)$, and the function $K_1(x)$ is given by

$$K_1(x) = \frac{1+x-e^{-x}}{x+\sinh(x)}. \tag{7}$$

Displacement-traction relation (6) provides a method for deriving the integral equations for the single punch normal indentation problem. From the definition of a Hankel inverse transform (Sneddon, 1972), $p_n(r)$ can be written as a Hankel transform of $A_n(s)$

$$p_n(r) = \int_0^\infty A_n(s)sJ_n(rs)\,\mathrm{d}s. \tag{8}$$

By virtue of the identity

$$\int_0^\infty J_\alpha(xs)J_\beta(ys)s^{1+\beta-\alpha}\,\mathrm{d}s = \frac{2^{1+\beta-\alpha}y^\beta H(x-y)}{x^\alpha\Gamma(\alpha-\beta)(x^2-y^2)^{1+\beta-\alpha}}, \quad \text{for } \alpha > \beta > -1, \tag{9}$$

it can be shown that the normal stress boundary condition on $z = 0$, for the single punch normal indentation problem, which requires that pressure outside the contact region P be zero, is satisfied by choosing $A_n(s)$ to be of the form

$$A_n(s) = -2\mu\sqrt{s}\int_0^a \sqrt{t}X_n(t)J_{n-1/2}(st)\,\mathrm{d}t. \tag{10}$$

Here $\Gamma(x)$ is the Gamma function, $H(x)$ is the Heaviside step function and $X_n(t)$ is a function to be determined on $[0, a]$. Considering that $u_z^n(r,0)$ is given when $r \leqslant a$, the substitution of eqn (10) into (6) with a change in the order of integration leads an integral equation for $X_n(t)$. Reference to the normal indentation of an elastic layer by a single punch can be found in the work by Keer (1964).

It is evident that the equivalent solutions, in terms of coordinates $(\bar{r}, \bar{\theta}, \bar{z})$ to the normal indentation problem involving single punch $\bar{P}$ can be obtained by eqns (6) and (10) with r, u_z, A_n and X_n replaced by $\bar{r}$, $\bar{u}_z$, $\bar{A}_n$ and $\bar{X}_n$.

Now consider the two punch normal indentation problem. It is clear that a superposition of the solutions for two single punch (either P or $\bar{P}$) normal indentation problems satisfies all the boundary conditions except the displacement conditions on the top surface $z = 0$. It will be shown that these conditions result a system of coupled Fredholm integral equations for $X_n(t)$ and $\bar{X}_n(t)$. In deriving these integral equations, expressions of the type (6) and its analogue for punch $\bar{P}$ in both local co-ordinates are required. This task can be accomplished by using the same technique proposed in Graham and Lan (1994). In terms of the second system of local co-ordinates $(\bar{r}, \bar{\theta}, z)$, the equivalent to the relation (6) can be rewritten as

$$u_z^n(\bar{r}, 0) = -\frac{(1-\nu)}{\mu}\int_0^\infty A_n^*(s)[1-K_1(2sh)]J_n(\bar{r}s)\,\mathrm{d}s, \tag{11}$$

with

$$A_n^*(s) = (-1)^n \sum_{m=0}^{\infty}{}' A_m(s)T_{mn}^1(fs) \tag{12}$$

and

$$T_{mn}^1(fs) = J_{m+n}(fs) + (-1)^n J_{m-n}(fs). \tag{13}$$

Here the prime on the summation sign implies that the $(-1)^n J_{m-n}(fs)$ terms do not appear when $n = 0$. In terms of the first local co-ordinate system, expression of the displacement-traction relation for the problem involving the second punch $\bar{P}$ takes the form

$$\bar{u}_z^n(r, 0) = -\frac{(1-\nu)}{\mu}\int_0^\infty \bar{A}_n^*(s)[1-K_1(2sh)]J_n(rs)\,\mathrm{d}s, \tag{14}$$

with

$$\bar{A}_n^*(s) = \sum_{m=0}^{\infty}{}'(-1)^m A_m(s)T_{mn}^1(fs). \tag{15}$$

Equations (11) and (14) are results of special importance. Superposing the above two normal displacement fields in the two local coordinate systems respectively and substituting them into the normal displacement boundary conditions (3), results in the following system of coupled Fredholm integral equations for $X_n(y)$ and $\bar{X}_n(y)$; i.e.

$$X_n(y) - \int_0^a X_n(t)K_{bn}(t, y)\,\mathrm{d}t + \sum_{m=0}^{\infty}{}'(-1)^m \int_0^{\bar{a}} \bar{X}_m(t)K_{pnm}(t, y)\,\mathrm{d}t = F_n(y), \quad \text{on } P, \tag{16}$$

$$\bar{X}_n(y) - \int_0^{\bar{a}} \bar{X}_n(t)K_{bn}(t, y)\,\mathrm{d}t + (-1)^n \sum_{m=0}^{\infty}{}' \int_0^a X_m(t)K_{pnm}(t, y)\,\mathrm{d}t = \bar{F}_n(y), \quad \text{on } \bar{P}. \tag{17}$$

Here the expression on the right hand sides of the integral eqn (16) is given by

$$F_n(y) = \frac{1}{2(1-\nu)}\sqrt{\frac{2}{\pi}}\frac{1}{y^n}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{x^{n+1}f_n(x)\,\mathrm{d}x}{\sqrt{y^2-x^2}} \tag{18}$$

and the expression for $\bar{F}_n(y)$ is identical to (18) except that $f_n(x)$ is replaced by $\bar{f}_n(x)$. $K_{bn}(t, y)$ are the kernel functions related to the boundary and $K_{pnm}(r, x)$ are the kernels reflecting the effects of the second punch (see the Appendix). For special cases, these integral equations can be solved by iteration. Once $X_n(t)$ and $\bar{X}_n(t)$ are known, pressures in the contact regions can be obtained by using eqn (8) and its analogue for $\bar{P}$. For example, pressure $p(r,\theta)$ under punch P takes the following form

$$p(r,\theta) = 2\mu\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty} r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{X_n(t)\,\mathrm{d}t}{t^{n-1}\sqrt{t^2-r^2}}\cos(n\theta). \tag{19}$$

The total force P_z in the z direction exerted by punch P can be obtained by integrating the pressure over the contact area P

$$P_z = -\int_0^{2\pi}\int_0^a p(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta = 4\mu\sqrt{2\pi}\int_0^a X_0(t)\,\mathrm{d}t, \tag{20}$$

and the resultant moment M_y in y direction can also be found from the result

$$M_y = \int_0^{2\pi}\int_0^a p(r,\theta)r^2\cos(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = -4\mu\sqrt{2\pi}\int_0^a tX_1(t)\,\mathrm{d}t. \tag{21}$$

2.2. *Some asymptotic solutions*

First we consider the single punch normal indentation problem, which has been studied by Keer (1964) using Copson's method for solving dual integral equations. From the kernels given in the Appendix, it is seen that due to the absence of the second punch, $K_{pnm}(t,y)$ reduces to zero and integral eqns (16) and (17) reduce to a system of integral equations for $X_n(t)$,

$$X_n(y) - \int_0^a X_n(t)K_n(t,y)\,\mathrm{d}t = F_n(y). \tag{22}$$

This recovers the result given by Keer (1964), noting that the definitions of the unknown functions are slightly different.

If the radius of the punch is small compared with the height of the layer, integral eqn (22) can be solved by iteration to obtain a solution perturbing the result for the problem of the normal indentation of a half-space. Note that kernels $K_n(x,y)$ can be expanded as power series in terms of a small non-dimensional parameter $\varepsilon_h = a/h$. Consequently the solutions for $X_n(t)$ can also be obtained in a similar form. For a slightly inclined punch with following displacement prescribed in contact area

$$u_z(r,\theta,0) = \delta_0+\delta_1 x = \delta_0+\delta_1 r\cos(\theta). \tag{23}$$

The eqn (22) can be solved by iteration; the solutions accurate to order $O(\varepsilon_h^3)$ are

$$X_0(y) = \frac{\delta_0}{(1-\nu)\sqrt{2\pi}}\left\{1+\frac{2}{\pi}L_1^0\varepsilon_h+\left(\frac{2}{\pi}L_1^0\right)^2\varepsilon_h^2+\left[\left(\frac{2}{\pi}L_1^0\right)^3-\frac{1}{\pi}(y^2+\tfrac{1}{3})L_1^2\right]\varepsilon_h^3\right\}, \tag{24}$$

$$X_1(y) = \frac{2\delta_1 y}{\sqrt{2\pi}(1-\nu)a}\left[1+\frac{2}{3\pi}L_1^2\varepsilon_h^3\right], \tag{25}$$

where L_1^0 and L_1^2 (given in the Appendix) are certain definite integrals involving function $K_1(x)$. These integrals can be evaluated by numerical integration with sufficient accuracy using the Gauss-Laguerre quadrature formula. The resultant force and moment then follow from eqns (20) and (21)

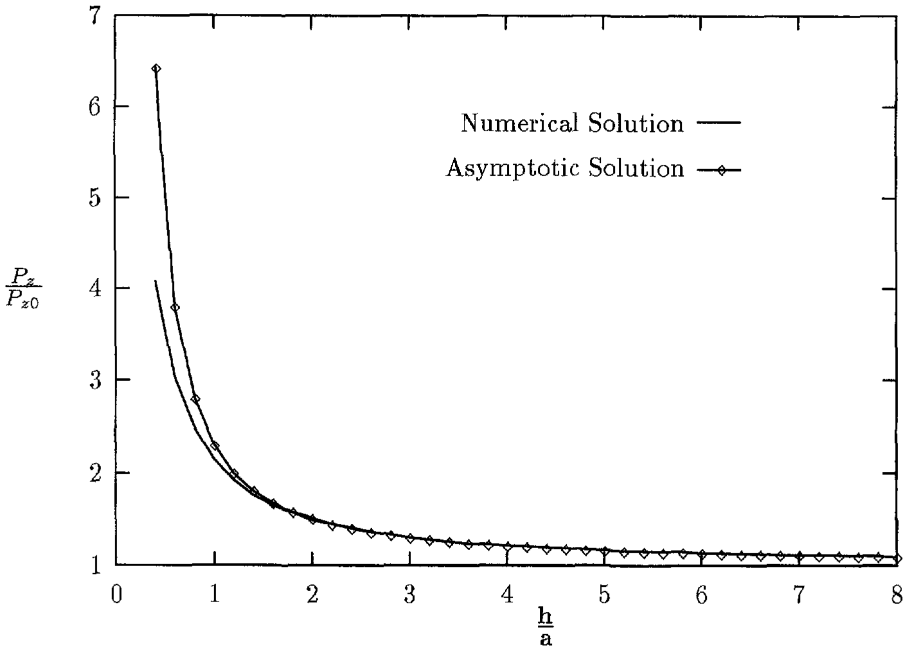

Fig. 2. Numerical and asymptotic solutions for the single punch normal indentation problem.

$$P_z = -\frac{4a\mu\delta_0}{(1-v)}\{1+0.7433\varepsilon_h+0.5525\varepsilon_h^2+0.075\varepsilon_h^3\}, \tag{26}$$

$$M_y = -\frac{8\mu a^2\delta_1}{3(1-v)}[1+0.3356\varepsilon_h^3]. \tag{27}$$

The above results show that P_z is only dependent on $f_0(x)$, the first Fourier coefficient of the given displacement, and M_y is fully determined by the second Fourier coefficient $f_1(x)$. These formulae, by definition, are valid to $O(\varepsilon_h^4)$. For moderate values of ε_h, eqn (22) can be solved directly by a numerical procedure. From Fig. 2 it is evident that the asymptotic solution gives a very good approximation to the normal indentation of the layer by a single punch if $h/a \geqslant 1$. In this figure $P_{z0} = -4a\mu\delta_0/(1-v)$ is the force that should be applied to a single punch on an elastic half-space, required to maintain a normal indentation δ_0.

As the second example, we consider the problem of two identical punches penetrating the elastic layer to equal depths δ_0; i.e. $f_0(x) = \bar{f}_0(x) = \delta_0$ and $f_n(x) = \bar{f}_n(x) = 0$ for $n \geqslant 1$. From the symmetry of this problem, it is seen that $X_n(t) = (-1)^n \bar{X}_n(t)$, and therefore the system of dual integral eqns (16) and (17) can again be reduced to a system of integral equations for $X_n(t)$,

$$X_n(y) - \int_0^a X_n(t) K_{bn}(t,y)\,dt + \sum_{m=0}^{\infty} \int_0^a X_m(t) K_{pnm}(t,y)\,dt = F_n(y). \tag{28}$$

If the punch radius a is small compared with either the thickness of the layer h or the distance f between the centers of the two punches, the above integral equation can be solved by iteration to obtain solutions in terms of double power series of ε_h and $\varepsilon_f(=a/f)$, which are power series in ε_f with coefficients being power series of ε_h. From the definitions of kernels given in the Appendix, it is seen that $K_{bn}(t,y)$ can be expanded as power series in ε_h of order $(2n+1)$ and $K_{pnm}(t,y)$ can be expanded as power series in ε_f of order $(m+n+1)$. If we seek solutions accurate to order $O(\varepsilon_h^2)+O(\varepsilon_f^2)$, all the equations corresponding to the terms $n \geqslant 2$ and $m \geqslant 2$ can be ignored and we only need to consider the first two integral equations for $X_0(t)$ and $X_1(t)$. The solutions are

$$X_0(y) = \frac{\delta_0}{(1-\nu)\sqrt{2\pi}}\left\{\left[1+\frac{2}{\pi}L_1^0\varepsilon_h+\left(\frac{2}{\pi}L_1^0\right)^2\varepsilon_h^2\right]+\frac{2}{\pi}(J_{01}^0-1)\left[1+\frac{4}{\pi}L_1^0\varepsilon_h+\frac{12}{\pi^2}(L_1^0)^2\varepsilon_h^2\right]\varepsilon_f\right.$$
$$\left.+\frac{4}{\pi^2}(J_{01}^0-1)^2\left[1+\frac{6}{\pi}L_1^0\varepsilon_h+\frac{24}{\pi^2}(L_1^0)^2\varepsilon_h^2\right]\varepsilon_f^2\right\}, \quad (29)$$

$$X_1(y) = \frac{4y}{\pi a}(J_{11}^1-1)\frac{\delta_0}{(1-\nu)\sqrt{2\pi}}\left[1+\frac{2}{\pi}L_1^0+\left(\frac{2}{\pi}L_1^0\right)^2\varepsilon_h^2\right]\varepsilon_f^2, \quad (30)$$

where J_{m1}^n (see Appendix) are certain definite integrals involving the function $K_1(x)$ and the spacing ratio h/f. The resultant force and moment applied on the first punch can then be obtained from (20) and (21) as follows:

$$P_z = -\frac{4a\mu\delta_0}{(1-\nu)}\left\{\left[1+\frac{2}{\pi}L_1^0\varepsilon_h+\left(\frac{2}{\pi}L_1^0\right)^2\varepsilon_h^2\right]+\frac{2}{\pi}(J_{01}^0-1)\left[1+\frac{4}{\pi}L_1^0\varepsilon_h+\frac{12}{\pi^2}(L_1^0)^2\varepsilon_h^2\right]\varepsilon_f\right.$$
$$\left.+\frac{4}{\pi^2}(J_{01}^0-1)^2\left[1+\frac{6}{\pi}L_1^0\varepsilon_h+\frac{24}{\pi^2}(L_1^0)^2\varepsilon_h^2\right]\varepsilon_f^2\right\}, \quad (31)$$

$$M_y = -\frac{16\mu\delta_0 a^2(J_{11}^1-1)}{3\pi(1-\nu)}\left[1+\frac{2}{\pi}L_1^0\varepsilon_h+\left(\frac{2}{\pi}L_1^0\right)^2\varepsilon_h^2\right]\varepsilon_f^2. \quad (32)$$

In the special case when ε_h approaches zero, the result (31) reduces to that given by Collins (1963). Equation (32) shows that when the circular punches interact an extra moment M_y of order $O(\varepsilon_f^2)$ is required to maintain the uniform indentation δ_0. Generally M_y is relatively small (of order $O(\varepsilon_f^2)$), and therefore only the dependence of P_z on h/f is shown graphically in Fig. 3.

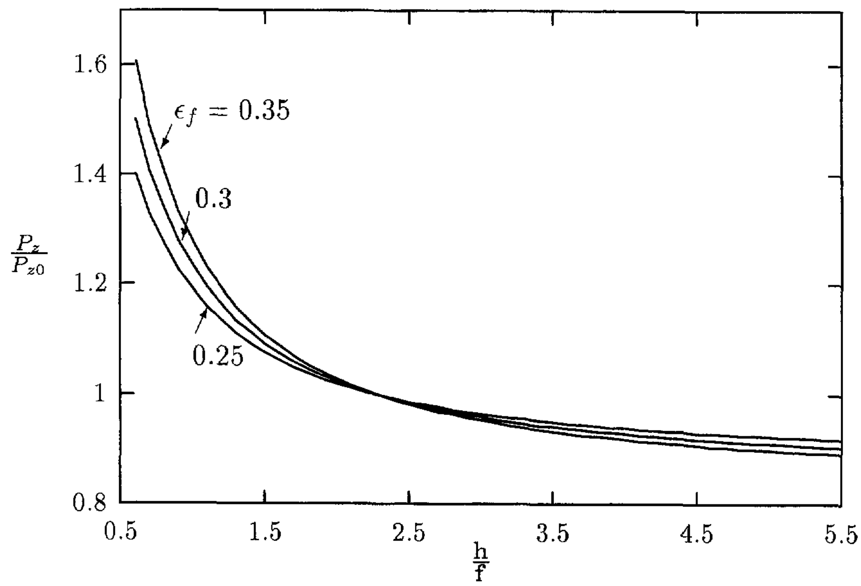

Fig. 3. Normal indentation: variation of the nondimensional resultant force P_z with respect to h/f for various ε_f.

Finally we use eqns (16) and (17) to examine the interaction between the rigid punch P and an externally located concentrated normal force $\bar{P}_z$. In this case, radius of the second punch $\bar{a} \to 0$ in such a way that $4\mu\sqrt{2\pi}\int_0^{\bar{a}} \bar{X}_0(t)\,\mathrm{d}t = \bar{P}_z$. We evaluate the force resultant and the moment required to maintain zero normal displacement under the punch P. For this limiting case, integral eqns (16) and (17) decouple. $X_n(t)$, $n = 0, 1, 2\ldots$ can be obtained by solving (16), which now takes the following form

$$X_n(y) - \int_0^a X_n(t) K_{bn}(t, y)\,\mathrm{d}t = -\frac{\bar{P}_z}{4\mu\sqrt{2\pi}} K_{pn0}(0, y). \tag{33}$$

As with the last two examples this equation can be solved by iteration. The first two terms of the power series solutions for quantities of interest are as follows

$$X_0(t) = -\frac{\bar{P}_z}{2\pi\sqrt{2\pi}\mu}\left[\frac{1}{\sqrt{f^2 - y^2}} + \frac{1}{a}\left(\frac{2}{\pi} L_1^0 \arcsin\frac{a}{f} - I_0^0\right)\varepsilon_h\right], \tag{34}$$

$$X_1(t) = -\frac{\bar{P}_z}{\mu\pi\sqrt{2\pi}}\left[\frac{t}{f\sqrt{f^2 - t^2}} - \frac{t}{a^2} I_1^1 \varepsilon_h^2\right], \tag{35}$$

where I_m^n are integrals involving function $K(x)$ and a parameter f/h (see Appendix). Then the force and moment required to maintain the zero normal displacement follow from (20) and (21)

$$P_z = \frac{2\bar{P}_z}{\pi}\left[\arcsin\left(\frac{a}{f}\right) + \left(\frac{2}{\pi} L_1^0 \arcsin\frac{a}{f} - I_0^0\right)\varepsilon_h\right], \tag{36}$$

$$M_y = \frac{2}{\pi}\bar{P}_z f\left[\arcsin\left(\frac{a}{f}\right) - \frac{a}{f}\sqrt{1 - \frac{a^2}{f^2}} - \frac{2a}{3f} I_1^1 \varepsilon_h^2\right]. \tag{37}$$

These results reduce to those given by Selvadurai (1980) when $\varepsilon_h \to 0$. It is evident that the above two expressions can be used to study interaction between the rigid punch P and an externally located concentrated moment M by applying two forces $\bar{P}_z$ and $-\bar{P}_z$ at $(f, 0, 0)$ and $(f+\delta, 0, 0)$ respectively and letting $\delta \to 0$ in such a way that $\bar{P}_z\delta = M$. Furthermore it is worth mentioning that these two results can also be obtained by using Betti's reciprocal theorem (Selvadurai, 1981).

3. TANGENTIAL INDENTATION PROBLEM

For the tangential indentation problem, the layer $0 \leqslant z \leqslant h$ is bonded with the rigid base at $z = h$ and the punches P and $\bar{P}$ on the layer are flexible so that there is no normal stress induced on the indentation surface $z = 0$ (see Figs 4 and 5).

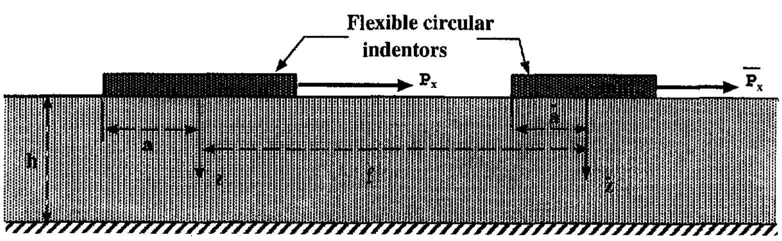

Fig. 4. Tangential indentation of a layer by two flexible circular indentors.

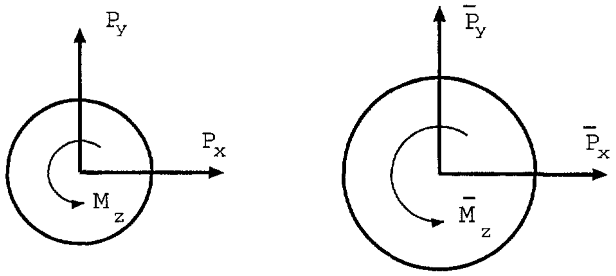

Fig. 5. Tangential indentation: upper surface of the elastic layer.

We assume that the tangential displacements are prescribed in the contact areas and can be expanded as Fourier series. Again our task is to relate the resultant shear forces required to maintain the given indentations.

In terms of the two co-ordinate systems introduced before, the boundary conditions of the tangential indentation problem can be written as

$$u_r^n(r,0)+u_\theta^n(r,0) = g_n(r), \quad u_r^n(r,0)-u_\theta^n(r,0) = h_n(r), \quad \text{for } r \leqslant a,$$
$$u_{\bar r}^n(\bar r,0)+u_{\bar\theta}^n(\bar r,0) = \bar g_n(\bar r), \quad u_{\bar r}^n(\bar r,0)-u_{\bar\theta}^n(\bar r,0) = \bar h_n(\bar r), \quad \text{for } \bar r \leqslant \bar a \tag{38}$$

and

$$\tau_{rz}^n(r,0) = \tau_{\theta z}^n(r,0) = 0, \quad \text{for} \quad r > a \quad \text{or} \quad \bar r > \bar a \quad \text{and} \quad \sigma_z^n(r,0) = 0, \quad \text{for} \quad r \geqslant 0, \tag{39}$$

on the indentation surface $z = 0$, and in terms of co-ordinates (r, θ, z)

$$u_r^n(r,h) = u_\theta^n(r,h) = u_z^n(r,h) = 0, \tag{40}$$

at the base of the layer $z = h$. It may be noted that the displacements in the z-direction within the punch regions are unspecified. If the contact is fully bonded then the normal displacements within the punch region could be of the $\delta_1 x$-type indicated in (23). In this problem however, this displacement is left unspecified.

3.1. *Derivation of the integral equations*

As with the normal indentation problem, we first consider a layer problem, where an elastic layer $0 \leqslant z \leqslant h$ is subject to shear stresses at the upper surface $z = 0$, and the base of the layer $z = h$ is constrained from movement. In order to examine rotational displacements, we interchange the dependence in θ(even or odd) in (2). Again in the derivation of integral equations, we only require the relationships between the tangential displacements at the surface and and the associated applied shear forces. These relations can be obtained by using the results given by Muki (1961), i.e:

$$u_r^n(r,0)+u_\theta^n(r,0) = 2\int_0^\infty \{(1-\nu)[1+K_2(2sh)]B_n(s)-[1+K_3(2sh)]C_n(s)\}J_{n-1}(rs)\,\mathrm{d}s, \tag{41}$$

$$u_r^n(r,0)-u_\theta^n(r,0) = 2\int_0^\infty \{-(1-\nu)[1+K_2(2sh)]B_n(s)-[1+K_3(2sh)]C_n(s)\}J_{n+1}(rs)\,\mathrm{d}s, \tag{42}$$

for any $n \geqslant 0$, where functions $K_2(x)$ and $K_3(x)$ are given by

$$K_2(x) = \frac{-6+8\nu-[(3-4\nu)^2+(1-x)^2]e^x}{3-4\nu+[(3-4\nu)^2+1+x^2]e^x+(3-4\nu)e^{2x}}, \tag{43}$$

$$K_3(x) = -\frac{2}{1+e^x} \tag{44}$$

and functions $B_n(s)$, $C_n(s)$ are determined by Hankel transforms of the Fourier coefficients of the prescribed shear stresses $\tau^n_{\theta z}(r,0)$ and $\tau^n_{rz}(r,0)$ in the following way

$$C_n(s)-B_n(s) = \int_0^\infty [\tau^n_{rz}(r,0)+\tau^n_{\theta z}(r,0)]rJ_{n-1}(rs)\,\mathrm{d}s, \tag{45}$$

$$C_n(s)+B_n(s) = \int_0^\infty [\tau^n_{rz}(r,0)-\tau^n_{\theta z}(r,0)]rJ_{n+1}(rs)\,\mathrm{d}s. \tag{46}$$

As with the normal indentation problem, eqns (41)–(42) and (45)–(46) provide a means for deriving the integral equations governing the single punch tangential indentation problem. With the aid of the method proposed by Westmann (1965) for solving simultaneous pairs of integral equations, we can show that the stress boundary conditions on the indentation surface (which require that the shear stresses outside the contact region P be zero) are satisfied by choosing

$$B_0(s) = 0, \tag{47}$$

$$C_0(s) = \sqrt{s}\int_0^a \sqrt{t}Z_0(t)J_{1/2}(st)\,\mathrm{d}t, \tag{48}$$

$$B_n(s) = \sqrt{s}\int_0^a \sqrt{t}[Y_n(t)J_{n-3/2}(st)+\frac{1}{1-\nu}Z_n(t)J_{n+1/2}(st)]\,\mathrm{d}t, \tag{49}$$

$$C_n(s) = \sqrt{s}\int_0^a \sqrt{t}[-Y_n(t)J_{n-3/2}(st)+Z_n(t)J_{n+1/2}(st)]\,\mathrm{d}t. \tag{50}$$

The tangential displacement boundary conditions on the indentation surface give rise to a pair of integral equations for $Y_n(t)$ and $Z_n(t)$.

We now consider the two punch tangential indentation problem. Solutions to the problem can be considered as a superposition of solutions to two tangential indentation problems, each involving one punch (either P or $\bar{P}$). This superposition satisfies all the boundary conditions except the tangential displacement conditions (38) in the contact regions. By using the same procedure as for the normal indentation problem, it can be shown that these two conditions result in a system of four Fredholm integral equations for $Y_n(t)$, $Z_n(t)$ and $\bar{Y}_n(t)$, $\bar{Z}_n(t)$. Here $\bar{Y}_n(t)$, $\bar{Z}_n(t)$ are the analogues of $Y_n(t)$, $Z_n(t)$ for the second punch $\bar{P}$. For the punch P, the system of integral equations is

$$Z_0(y)+\int_0^a Z_0(t)K_{b022}(t,y)\,\mathrm{d}t+\sum_{m=1}^{\infty}(-1)^m\left\{\int_0^{\bar{a}} \bar{Y}_m(t)K_{p0m21}(t,y)\,\mathrm{d}t+\int_0^{\bar{a}} \bar{Z}_m(t)K_{p0m22}(t,y)\,\mathrm{d}t\right\}$$
$$+\int_0^{\bar{a}} \bar{Z}_0(t)K_{p0022}(t,y)\,\mathrm{d}t = -\frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}}{y\,\mathrm{d}y}\int_0^y \frac{x^2h_0(x)\,\mathrm{d}x}{\sqrt{y^2-x^2}}, \tag{51}$$

$$Y_0(y) = 0, \tag{52}$$

if $n = 0$ and

$$Y_n(y) + \int_0^a Y_n(t)K_{bn11}(t,y)\,\mathrm{d}t + \int_0^a Z_n(t)K_{bn12}(t,y)\,\mathrm{d}t + \sum_{m=1}^{\infty}(-1)^m\left\{\int_0^a \bar{Y}_n(t)K_{pnm11}(t,y)\,\mathrm{d}t + \int_0^a \bar{Z}_n(t)K_{pnm12}(t,y)\,\mathrm{d}t\right\} + \int_0^a \bar{Z}_0(t)K_{pn012}(t,y)\,\mathrm{d}t = G_n(y), \tag{53}$$

$$Z_n(y) + \int_0^a Y_n(t)K_{bn21}(t,y)\,\mathrm{d}t + \int_0^a Z_n(t)K_{bn22}(t,y)\,\mathrm{d}t + \sum_{m=1}^{\infty}(-1)^m\left\{\int_0^a \bar{Y}_n(t)K_{pnm21}(t,y)\,\mathrm{d}t + \int_0^a \bar{Z}_n(t)K_{pnm22}(t,y)\,\mathrm{d}t\right\} + \int_0^a \bar{Z}_0(t)K_{pn022}(t,y)\,\mathrm{d}t = H_n(y), \tag{54}$$

if $n \geqslant 1$. Here the quantities on right-hand side are

$$G_n(y) = \frac{y^{-n+1}}{(2-\nu)\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{x^n g_n(x)\,\mathrm{d}x}{\sqrt{y^2-x^2}}, \tag{55}$$

$$H_n(y) = -\frac{y^{-n-1}}{2\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{x^{n+2}h_n(x)\,\mathrm{d}x}{\sqrt{y^2-x^2}} + \frac{\nu}{2}\int_0^a Y_n(t)\sqrt{yt}\,\mathrm{d}t\int_0^\infty J_{n-3/2}(st)J_{n+1/2}(ys)s\,\mathrm{d}s, \tag{56}$$

and kernels of the integral equations are listed in the Appendix. Corresponding integral equations for the punch $\bar{P}$ take a similar form. In some instances, it is helpful to write the second term on the right hand side of eqn (56) in the alternative form, i.e.

$$-\frac{\nu}{2}\sqrt{y}\int_0^\infty J_{n+1/2}(ys)\,\mathrm{d}s\int_0^a t^{n-1/2}J_{n-1/2}(st)\frac{\mathrm{d}}{\mathrm{d}t}[Y_n(t)t^{1-n}]\,\mathrm{d}t.$$

Integral eqns (51)–(54) and their analogues for punch $\bar{P}$ can be solved when a is small compared with both h and f. Once $Y_n(t)$, $Z_n(t)$ are known, the total resultant force P_x, P_y and moment M_z can be obtained. Equations (45) and (46) enable us to find the stress distribution under punch P,

$$\tau^n_{\theta z}(r,0) + \tau^n_{rz}(r,0) = 4\sqrt{\frac{2}{\pi}}\mu r^{n-2}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{Y_n(t)\,\mathrm{d}t}{t^{n-2}\sqrt{t^2-r^2}} - \frac{2\mu\nu}{1-\nu}\sqrt{\frac{2}{\pi}}r^{-n}\frac{\mathrm{d}}{\mathrm{d}r}r^{2n}\int_r^a \frac{Z_n(t)\,\mathrm{d}t}{t^n\sqrt{t^2-r^2}}, \tag{57}$$

$$\tau_{\theta z}^{n}(r,0)-\tau_{rz}^{n}(r,0)=\frac{2\mu(2-\nu)}{1-\nu}\sqrt{\frac{2}{\pi}}r^{n}\frac{\mathrm{d}}{\mathrm{d}r}\int_{r}^{a}\frac{Z_{n}(t)\,\mathrm{d}t}{t^{n}\sqrt{t^{2}-r^{2}}}. \tag{58}$$

Proper integrations over the region P give the resultant forces and moment

$$P_{x}=\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{a}r\,\mathrm{d}r\tau_{zx}(r,\theta,0)=0, \tag{59}$$

$$P_{y}=\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{a}r\,\mathrm{d}r\tau_{zy}(r,\theta,0)=4\sqrt{2\pi}\mu\int_{0}^{a}Y_{1}(t)\,\mathrm{d}t, \tag{60}$$

$$M_{z}=\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{a}r\,\mathrm{d}r[r\tau_{\theta z}(r,\theta,0)]=4\mu\sqrt{2\pi}\int_{0}^{a}r^{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\int_{r}^{a}\frac{Z_{0}(t)\,\mathrm{d}t}{\sqrt{t^{2}-r^{2}}}\right]\mathrm{d}r. \tag{61}$$

Equation (59) shows that the total force in the x-direction vanishes, even though $\tau_{xz}(r,\theta,0)\neq 0$ in the contact area. Note that (59) and (60) hold only for the case in which $u_{\theta}(r,\theta,z)$ is an even function of θ, $u_{r}(r,\theta,z)$ and $u_{z}(r,\theta,z)$ are odd functions of θ. For the case where the θ dependence is given by (2), P_{y} is always zero as shown in the next subsection. In any case, force and moments in other directions, i.e. P_{z}, M_{x} and M_{y} are all zero due to the relaxation boundary condition $\sigma_{z}^{n}(r,0)=0$, for $r\geqslant 0$. However, to satisfy the relaxation condition, $u_{z}^{n}(r,0)$ is generally non-zero and punches have to be flexible to allow such non-uniform normal displacement. Expression for $u_{z}^{n}(r,0)$ in a special case is given in the next subsection.

3.2. *Some asymptotic solutions*

In this subsection integral eqns (51)–(54) are solved for some special cases.

First we consider the case where two identical circular punches of radius a and center to center spacing f which are subjected to equal rotational displacements (in the same or opposite direction). The boundary conditions can be expressed as

$$u_{\theta}^{0}(t,z)=\pm\bar{u}_{\theta}^{0}(t,z)=\delta_{0}t,\quad\text{and}\quad u_{\theta}^{n}(t,z)=\bar{u}_{\theta}^{n}(t,z)=0,\quad\text{for } n\neq 0. \tag{62}$$

Here the upper signs correspond to rotations in the same sense and the lower signs correspond to rotations in the opposite sense. It is seen, from the symmetry of this problem, that $Y_{n}(t)$, $Z_{n}(t)$ and $\bar{Y}_{n}(t)$, $\bar{Z}_{n}(t)$ satisfy the following

$$Y_{n}(t)=\pm(-1)^{n}\bar{Y}_{n}(t),\quad Z_{n}(t)=\pm(-1)^{n}\bar{Z}_{n}(t). \tag{63}$$

Therefore the system of four integral equations, two for each indentor, can be reduced to a system of two integral equations for $Y_{n}(t)$ and $Z_{n}(t)$. When a $(=\bar{a})$ is small compared with h and f, these equations can be solved iteratively to give solutions in terms of ε_{h} and ε_{f}. Solutions accurate to order $O(\varepsilon_{h}^{3})+O(\varepsilon_{f}^{3})$ are

$$Z_{0}(ay)=-\sqrt{\frac{2}{\pi}}\theta_{0}y\left[1-\frac{2}{3\pi}L_{3}^{2}\varepsilon_{h}^{3}\right]\left[1\mp\frac{2}{3\pi}(-1+J_{03}^{2})\varepsilon_{f}^{3}\right], \tag{64}$$

$$Y_{1}(ay)=\pm\sqrt{\frac{2}{\pi}}\theta_{0}\frac{1+J_{13}^{1}}{(2-\nu)\pi}\left\{1-\frac{2}{(2-\nu)\pi}[(1-\nu)L_{2}^{0}+L_{3}^{0}]\varepsilon_{h}\right\}\varepsilon_{f}^{2}, \tag{65}$$

where J_{ni}^{m} (given in the Appendix) are definite integrals with parameter h/f. Substituting

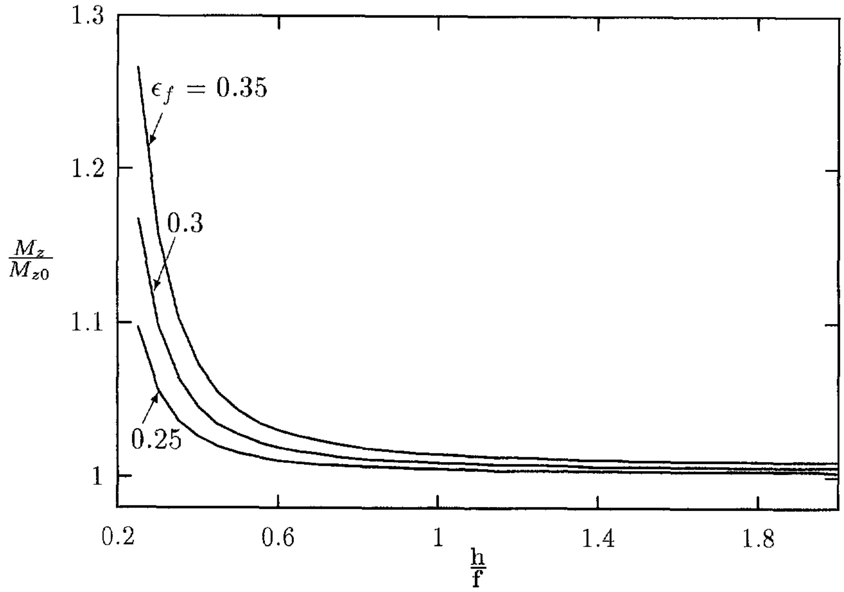

Fig. 6. Tangential indentation: variation of the nondimensional resultant moment M_z with respect to h/f for various ε_f in the case where two punches undergo equal rotational displacements (in same direction).

these solutions into eqns (61) and (60) gives us the following moment and force required to maintain the displacement (62)

$$M_z = \tfrac{16}{3}\mu a^3\delta_0\left[1-\frac{2}{3\pi}L_3^2\varepsilon_h^3\right]\left[1\mp\frac{2}{3\pi}(-1+J_{03}^2)\varepsilon_f^3\right], \tag{66}$$

$$P_y = \pm 8\mu\delta_0\frac{1+J_{13}^1}{(2-\nu)\pi}\left\{1-\frac{2}{(2-\nu)\pi}[(1-\nu)L_2^0+L_3^0]\varepsilon_h\right\}\varepsilon_f^2. \tag{67}$$

The last equation indicates that an extra y-direction force of order $O(\varepsilon_f^2)$ is needed to maintain the pure rotational displacement. The variations in the nondimensional M_z are shown in Fig. 6. These results are for Poisson's ratio $\nu = 0.3$.

As the second example, we consider two identical punches which undergo equal displacements $u_y = \pm\bar{u}_{\bar{y}} = \delta_0$ in the y-direction. We find again from the symmetry of this problem that solutions of the integral equations satisfy $Y_n(t) = \pm(-1)^{n+1}\bar{Y}_n(t)$ and $Z_n(t) = \pm(-1)^{n+1}\bar{Z}_n(t)$. The first few leading terms of the solutions produce the following resultant force and moment

$$P_y = \frac{8\mu a\delta_0}{2-\nu}\left\{1-\frac{2}{(2-\nu)\pi}[(1-\nu)L_2^0+L_3^0]\varepsilon_h\right\}$$
$$\times\left\{1\mp\frac{2}{(2-\nu)\pi}[2\nu-2+(\nu-1)(J_{22}^0+J_{02}^0)+J_{23}^0-J_{03}^0]\varepsilon_f\right\}, \tag{68}$$

$$M_z = \mp\frac{16\mu\delta_0(1+J_{31}^1)a^3}{3(2-\nu)\pi}\left\{1-\frac{2}{(2-\nu)\pi}[(1-\nu)L_2^0+L_3^0]\varepsilon_h\right\}\varepsilon_f^2. \tag{69}$$

Finally we consider the case where the layer is subjected to displacements $u_x = \pm\bar{u}_{\bar{x}} = \delta_0$ inside the contact areas P and $\bar{P}$. In this case, the θ dependence of the displacements changes back to (2) and therefore $Z_0(s) = \bar{Z}_0(s) = 0$ prevail instead of $Y_0(s) = \bar{Y}_0(s) = 0$. Integral eqns (51)–(54) only require some minor changes and the solutions are as follows

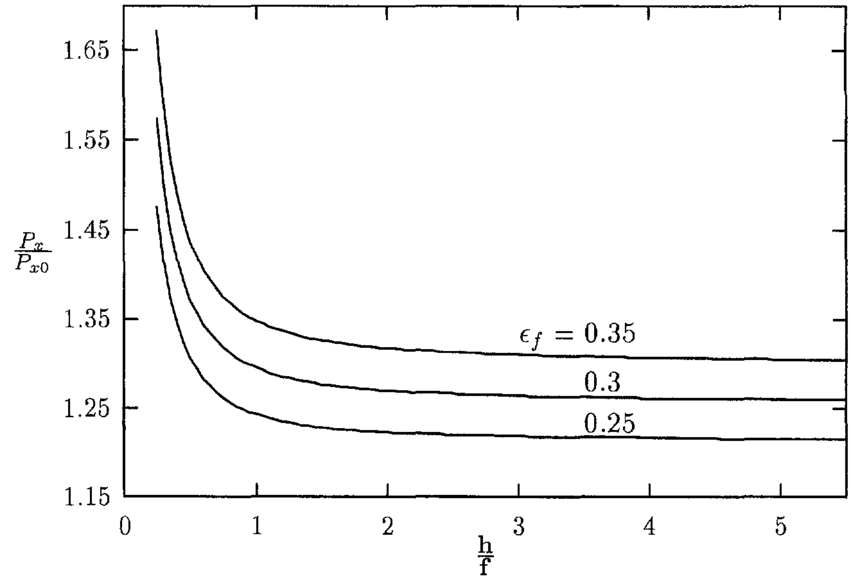

Fig. 7. Tangential indentation: variation of the nondimensional resultant force P_x with respect to h/f for various ε_f in the case where two punches undergo same x-direction displacements.

$$P_x = \frac{2\delta_0}{(2-\nu)\sqrt{2\pi}}\left\{1-\frac{2}{(2-\nu)\pi}[(1-\nu)L_2^0+L_3^0]\varepsilon_h\right\}$$

$$\times\left\{1\mp\frac{2}{(2-\nu)\pi}[-2+(1-\nu)(J_{22}^0-J_{02}^0)-J_{23}^0-J_{03}^0]\varepsilon_f\right\}, \quad (70)$$

and the resultant force in the y-direction, P_y, is always zero though $\tau_{zy} \neq 0$ inside the contact area P. Equations (68) and (70) agree with those given in Fabrikant (1989) to the order of $O(\varepsilon_f^2)$ when ε_h reduces to zero. The behavior of the P_x and P_y for nonzero ε_h are shown in Figs 7 and 8. Again, these results are applicable for the case $\nu = 0.3$. As mentioned before the normal displacements under the punches are generally non-zero. If a half-space is

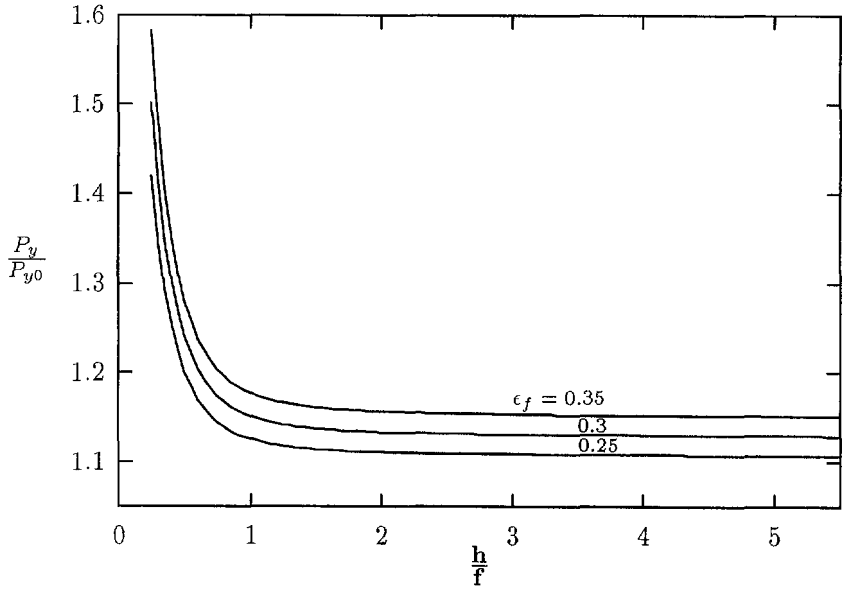

Fig. 8. Tangential indentation: variation of the nondimensional resultant force P_y with respect to h/f for various ε_f in the case where two punches undergo same y-direction displacements.

subjected to a displacement δ in the x-direction in the contact region P, then the absence of the normal stress in the contact surface results in a normal displacement. Under this flexible punch this displacement is as follows

$$u_z(r,\theta,0) = \frac{2(1-2\nu)\delta}{(2-\nu)\pi}\frac{r\sin(\theta)}{(a+\sqrt{a^2-r^2})}. \tag{71}$$

This result is in keeping with the assertion which follows (40).

4. CONCLUSIONS

The paper demonstrates the basic formulations which govern the combined indentation of a layer underlain by a rigid base, by two circular punches. In the case of either the smooth axial indentation or the relaxed in-plane (tangential) indentation the formulation results in systems of Fredholm integral equations. The paper develops numerical procedures which can be used to find approximate solutions to problems of engineering interest. In particular the approximate results developed via asymptotic method are compared with available limiting results for punches located on half-space regions. The methodology can be extended quite conveniently to include other types of indentations and transverse isotropy of the elastic layer. These extensions will be considered in future work.

Acknowledgement—Q. Lan would like to acknowledge the Post-Doctoral Fellowship awarded by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

Collins, W. D. (1963). Some coplanar punch and crack problems in three-dimensional elastostatics. *Proc. Roy. Soc. London* **A274,** 507–528.
Fabrikant, V. I. (1986). Several elliptical punches on an elastic half-space. *J. Appl. Mech.* **53,** 390–394.
Fabrikant, V. I. (1989). Elastic half-space with several domains subjected to tangential displacements. *Ingenieur-Archiv* **60,** 73–82.
Gladwell, G. M. L. and Fabrikant, V. I. (1982). The interaction between a system of circular punches on an elastic half space. *J. Appl. Mech.* **49,** 341–344.
Graham, G. A. C. and Lan, Q. (1994). Stress intensity factor for two offset parallel circular cracks: Part I—Infinite solid. *Theo. Appl. Frac. Mech.* **20,** 207–225.
Keer, L. M. (1964). A class of non-symmetrical punch and crack problems. *Quart. J. Mech. Appl. Math.* **17,** 423–436.
Muki, R. (1961). Asymmetric problems of the theory of elasticity for a semi-infinite solid and a thick plate. In *Progress in Solid Mechanics* Vol. 1 (eds I. N. Sneddon and R. Hill). North-Holland Publishing Company, Armsterdam, pp. 401–439.
Selvadurai, A. P. S. (1980). On the displacement induced in a rigid circular punch on an elastic halfspace due to an external force. *Mech. Research Comm.* **7,** 351–358.
Selvadurai, A. P. S. (1981). Betti's reciprocal relationships for the displacement of an elastic infinite space bounded internally by a rigid inclusion. *J. Struct. Mech.* **9,** 199–210.
Sneddon, I. N. (1972). *The Use of Integral Transforms*, McGraw-Hill, New York.
Westmann, R. A. (1965). Simultaneous pairs of dual integral equations. *SIAM Review* **7,** 341–348.

APPENDIX

Kernels of integral eqns (16) and (17) are

$$K_{bn}(t,y) = \sqrt{yt}\int_0^\infty J_{n-1/2}(st)J_{n-1/2}(ys)K_1(2sh)s\,\mathrm{d}s,$$

$$K_{pnm}(t,y) = \sqrt{yt}\int_0^\infty J_{m-1/2}(st)J_{n-1/2}(ys)T^1_{mn}(fs)s[1-K_1(2sh)]\,\mathrm{d}s.$$

Kernels of integral eqns (51)–(54) are

$$K_{b022}(t,y) = \sqrt{yt}\int_0^\infty K_3(2sh)J_{1/2}(st)J_{1/2}(ys)s\,\mathrm{d}s,$$

$$K_{p0022}(t,y) = \sqrt{yt}\int_0^\infty [1+K_3(2sh)]J_{1/2}(st)J_{1/2}(ys)J_0(fs)s\,\mathrm{d}s,$$

$$K_{p0m21}(t,y) = -\sqrt{yt}\int_0^\infty J_{1/2}(ys)J_{m-3/2}(st)J_m(fs)[1+K_3(2sh)]s\,\mathrm{d}s,$$

$$K_{p0m22}(t,y) = \sqrt{yt}\int_0^\infty J_{1/2}(ys)J_{m+1/2}(st)J_m(fs)[1+K_3(2sh)]s\,\mathrm{d}s,$$

$$K_{bn11}(t,y) = \frac{\sqrt{yt}}{2-v}\int_0^\infty [(1-v)K_2(2sh)+K_3(2sh)]J_{n-3/2}(st)J_{n-3/2}(ys)s\,\mathrm{d}s,$$

$$K_{bn12}(t,y) = \frac{\sqrt{yt}}{2-v}\int_0^\infty [K_2(2sh)-K_3(2sh)]J_{n+1/2}(st)J_{n-3/2}(ys)s\,\mathrm{d}s,$$

$$K_{bn21}(t,y) = \frac{\sqrt{yt}}{2}\int_0^\infty [(1-v)K_2(2sh)-K_3(2sh)]J_{n-3/2}(st)J_{n+1/2}(ys)s\,\mathrm{d}s,$$

$$K_{bn22}(t,y) = \frac{\sqrt{yt}}{2}\int_0^\infty [K_2(2sh)+K_3(2sh)]J_{n+1/2}(st)J_{n+1/2}(ys)s\,\mathrm{d}s,$$

$$K_{pn012}(t,y) = -\frac{2\sqrt{yt}}{2-v}\int_0^\infty [(1+K_3(2sh)]J_{1/2}(st)J_{n-3/2}(ys)J_n(fs)s\,\mathrm{d}s,$$

$$K_{pn022}(t,y) = \sqrt{yt}\int_0^\infty [1+K_3(2sh)]J_{1/2}(st)J_{n+1/2}(ys)J_n(fs)s\,\mathrm{d}s,$$

$$K_{pnm11}(t,y) = \frac{\sqrt{yt}}{2-v}\int_0^\infty [(1-v)R_2(s)+R_3(s)]J_{m-3/2}(st)J_{n-3/2}(ys)s\,\mathrm{d}s,$$

$$K_{pnm12}(t,y) = \frac{\sqrt{yt}}{2-v}\int_0^\infty [R_2(s)-R_3(s)]J_{m+1/2}(st)J_{n-3/2}(ys)s\,\mathrm{d}s,$$

$$K_{pnm21}(t,y) = \frac{\sqrt{yt}}{2}\int_0^\infty [(1-v)R_2(s)-R_3(s)]J_{m-3/2}(st)J_{n+1/2}(ys)s\,\mathrm{d}s,$$

$$K_{pnm22}(t,y) = \frac{\sqrt{yt}}{2}\int_0^\infty [R_2(s)+R_3(s)]J_{n+1/2}(st)J_{n+1/2}(ys)s\,\mathrm{d}s,$$

where

$$R_2(s) = -T^2_{mn}(fs)[1+K_2(2sh)], \quad R_3(s) = T^1_{mn}(fs)[1+K_3(2sh)],$$

and

$$T^2_{mn}(fs) = J_{m-n}(fs)-(-1)^n J_{m-n}(fs).$$

The following are some definite integrals involved in the asymptotic solutions

$$I^n_m = \int_0^\infty K_1(2x)J_m\left(\frac{f}{h}x\right)x^n\,\mathrm{d}x,$$

$$L^n_i = \int_0^\infty K_i(2x)x^n\,\mathrm{d}x,$$

$$J^n_{mi} = \int_0^\infty J_m(x)K_i\left(2x\frac{h}{f}\right)x^n\,\mathrm{d}x, \quad i = 1,2,3 \quad \text{and} \quad m,n\ \text{are integers.}$$

Here L^n_i are constants when $i = 1, 3$ and are functions of v when $i = 2$. For a given value of v, say $v = 0.3$, all L^n_i can be evaluated numerically using the Gauss-Laguerre formula with sufficient accuracy. I^n_m are functions of f/h and J^n_{mi} are functions of h/f only when $i = 1, 3$ and functions of h/f and v when $i = 2$. For given h/f and v, all I^n_m and J^n_{mi} can also be evaluated with sufficient accuracy using the shifted Gauss-Laguerre formula.